

# ASSOCIATED PRIMES OF POWERS OF EDGE IDEALS

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**ABSTRACT.** Let  $G$  be a graph and let  $I$  be its edge ideal. Our main result shows that the sets of associated primes of the powers of  $I$  form an ascending chain. It is known that the sets of associated primes of  $I^i$  and  $\bar{I}^i$  stabilize for large  $i$ . We show that their corresponding stable sets are equal. To show our main result we use a classical result of Berge from matching theory and certain notions from combinatorial optimization.

## 1. INTRODUCTION

Let  $G$  be a simple *graph* with finite vertex set  $X = \{x_1, \dots, x_n\}$ , i.e.,  $G$  is the set  $X$  together with a family of subsets of  $X$  of cardinality 2, called edges, none of which is included in another. The sets of vertices and edges of  $G$  are denoted by  $V(G)$  and  $E(G)$  respectively. We shall always assume that  $G$  has no isolated vertices, i.e., every vertex of  $G$  has to occur in at least one edge. Let  $R = K[x_1, \dots, x_n]$  be a polynomial ring over a field  $K$ . The *edge ideal* of  $G$ , denoted by  $I = I(G)$ , is the ideal of  $R$  generated by all square-free monomials  $x_i x_j$  such that  $\{x_i, x_j\} \in E(G)$ . The assignment  $G \mapsto I(G)$  gives a natural one to one correspondence between the family of graphs and the family of monomial ideals generated by square-free monomials of degree 2.

Let  $G$  be a graph and let  $I = I(G)$  be its edge ideal. In this paper we will examine the sets of associated primes of the powers of  $I$ , that is, the sets

$$\text{Ass}(R/I^k) = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is prime and } \mathfrak{p} = (I^k : c) \text{ for some } c \in R\}, \quad k \geq 1.$$

Since  $I$  is a monomial ideal of a polynomial ring  $R$ , the associated primes will be monomial primes, which are primes that are generated by subsets of the variables, see [26, Proposition 5.1.3]. The associated primes of  $I$  correspond to minimal vertex covers of the graph  $G$  and  $\text{Min}(R/I) = \text{Ass}(R/I)$ , where  $\text{Min}(R/I)$  denotes the set of minimal primes of  $I$ , see [25]. For edge ideals,  $\text{Ass}(R/I) \subset \text{Ass}(R/I^k)$  for all positive integers  $k$ . In the case where equality holds for all  $k$ , the ideal  $I$  is said to be *normally torsion-free*.

In [1], Brodmann showed that the sets  $\text{Ass}(R/I^k)$  stabilize for large  $k$ . That is, there exists a positive integer  $N_1$  such that  $\text{Ass}(R/I^k) = \text{Ass}(R/I^{N_1})$  for all  $k \geq N_1$ . A minimal such  $N_1$  is called the *index of stability* of  $I$ . One important result in this area establishes that  $N_1 = 1$  if and only if  $G$  is a bipartite graph [23, Theorem 5.9]. A useful upper bound for  $N_1$  was shown in [3, Corollary 4.3], namely that if  $G$  is a connected non-bipartite graph with  $n$  vertices,  $s$  leaves, and the smallest odd cycle of  $G$  has length  $2k + 1$ , then  $N_1 \leq n - k - s$ . We make use of this upper bound in Example 3.9.

Although the sets  $\text{Ass}(R/I^k)$  are known to stabilize for large  $k$ , their behavior for small  $k$  can be erratic. Finding the stable set  $\text{Ass}(R/I^{N_1})$  is complicated by the fact that a prime ideal

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$\mathfrak{p}$  that is associated to a low power of an ideal  $I$  need not be associated to higher powers. For example, [18, Example, p. 2] gives an example, due to A. Sathaye, of an ideal  $I$  in a ring  $R$  and a prime  $\mathfrak{p}$  for which  $\mathfrak{p} \in \text{Ass}(R/I^k)$  for  $k$  even and  $\mathfrak{p} \notin \text{Ass}(R/I^k)$  for  $k$  odd for all  $k$  below a stated bound. When, for an ideal  $I$ ,  $\mathfrak{p} \in \text{Ass}(R/I^k)$  implies  $\mathfrak{p} \in \text{Ass}(R/I^{k+1})$  for all  $k \geq 1$ , one says that the sets  $\text{Ass}(R/I^k)$  form an *ascending chain*. Although this property is highly desirable, few classes of ideals are known to possess it. Examples of monomial ideals for which the sets  $\text{Ass}(R/I^k)$  do not form ascending chains can be found in [12, Section 4] (stated in terms of depths), and [19, Example 4.18].

Let  $\overline{I^k}$  denote the integral closure of  $I^k$ . An ideal  $I$  is called *normal* if  $I^k = \overline{I^k}$  for all  $k \geq 1$ . By results of Ratliff [20, 21], one has that the sets  $\text{Ass}(R/\overline{I^k})$  form an ascending chain which stabilizes for large  $k$ . Thus, there exists  $N_2$  such that  $\text{Ass}(R/\overline{I^k}) = \text{Ass}(R/\overline{I^{N_2}})$  for  $k \geq N_2$ . The set  $\text{Ass}(R/\overline{I^{N_2}})$  is nicely described in [17], and for edge ideals of graphs the set  $\text{Ass}(R/I^{N_1})$  is described in [3].

Our main result is:

**Theorem 2.15** *If  $I$  is the edge ideal of a graph, then  $\text{Ass}(R/I^k) \subset \text{Ass}(R/I^{k+1})$  for all  $k$ . That is, the sets of associated primes of the powers of  $I$  form an ascending chain.*

There are two cases where the sets of associated primes of a square-free monomial ideal are known to form an ascending chain. The first case is the family of normal ideals (as was pointed out above), which includes, for instance, ideals of vertex covers of perfect graphs [7, 8, 27]. The second case is the family of graphs with at least one leaf [19], which is now a particular case of our main result.

In a more general setting, i.e., when  $I \neq (0)$  is an ideal of a commutative Noetherian domain, Ratliff showed that  $(I^{k+1} : I) = I^k$  for all large  $k$  [20, Corollary 4.2] and that equality holds for all  $k$  when  $I$  is normal [20, Proposition 4.7]. We show that equality holds for all  $k$  when  $I$  is an edge ideal.

**Lemma 2.12**  $(I^{k+1} : I) = I^k$  for  $k \geq 1$ .

This lemma is central to the proof of our main result. To show this lemma, we need to link the algebraic and combinatorial data. This is achieved using matching theory and basic notions from combinatorial optimization.

Given an edge  $f$ , we denote by  $G^f$  the graph obtained from  $G$  by duplicating the two vertices of  $f$  (see Definition 2.1). The *deficiency* of  $G$ , denoted by  $\text{def}(G)$ , is the number of vertices left uncovered by any maximum matching of  $G$ . The matching number of  $G$  is denoted by  $\nu(G)$  (see Definition 2.3). Using a formula of Berge (see Theorem 2.7), we compare the deficiencies of  $G$  and  $G^f$ .

Our main combinatorial result is:

**Theorem 2.8**  $\text{def}(G^f) = \delta$  for all  $f \in E(G)$  if and only if  $\text{def}(G) = \delta$  and  $\nu(G^f) = \nu(G) + 1$  for all  $f \in E(G)$ .

As a byproduct, we present the following characterization of graphs with a perfect matching.

**Corollary 2.11**  $G$  has a perfect matching if and only if  $G^f$  has a perfect matching for every edge  $f$  of  $G$ .

In general, for ideals in commutative Noetherian rings,  $\text{Ass}(R/\overline{I^{N_2}})$  is a subset of  $\text{Ass}(R/I^{N_1})$ , see [18, Proposition 3.17]. We show that for edge ideals these stable sets are equal.

**Theorem 3.6**  $\text{Ass}(R/I^k) = \text{Ass}(R/\overline{I^k})$  for  $k \geq \max\{N_1, N_2\}$ .

As an application we show that an edge ideal  $I$  is normally torsion-free if and only if  $\text{Ass}(R/\overline{I}^i) = \text{Ass}(R/I)$  for  $i \geq 1$  (see Corollary 3.7).

Throughout the paper we introduce most of the notions that are relevant for our purposes. For unexplained terminology we refer to [4, 16, 24]. Two excellent references for the general theory of asymptotic prime divisors in commutative Noetherian rings are [13] and [18].

## 2. PERFECT MATCHINGS AND PERSISTENCE OF ASSOCIATED PRIMES

In this section we give a characterization of graphs with a perfect matching and show that the sets of associated primes of powers of an edge ideal form an ascending chain. We continue using the definitions and terms from the introduction.

Let  $G$  be a graph with vertex set  $X = \{x_1, \dots, x_n\}$  and let  $I = I(G) \subset R$  be its edge ideal. In what follows  $F = \{f_1, \dots, f_q\}$  denotes the set of all monomials  $x_i x_j$  such that  $\{x_i, x_j\} \in E(G)$ . As usual, we use  $x^a$  as an abbreviation for  $x_1^{a_1} \cdots x_n^{a_n}$  and we set  $|a| = a_1 + \cdots + a_n$ , where  $a = (a_i) \in \mathbb{N}^n$ . For convenience we will consider 0 to be an element of  $\mathbb{N}$ . We also use  $f^c$  as an abbreviation for  $f_1^{c_1} \cdots f_q^{c_q}$ , where  $c = (c_i) \in \mathbb{N}^q$ .

**Definition 2.1.** Following Schrijver [22], the *duplication* of a vertex  $x_i$  of a graph  $G$  means extending its vertex set  $X$  by a new vertex  $x'_i$  and replacing  $E(G)$  by

$$E(G) \cup \{(e \setminus \{x_i\}) \cup \{x'_i\} \mid x_i \in e \in E(G)\}.$$

The *deletion* of  $x_i$ , denoted by  $G \setminus \{x_i\}$ , is the graph formed from  $G$  by deleting the vertex  $x_i$  and all edges containing  $x_i$ . A graph obtained from  $G$  by a sequence of deletions and duplications of vertices is called a *parallelization* of  $G$ .

It is not difficult to verify that these two operations commute. If  $a = (a_i)$  is a vector in  $\mathbb{N}^n$ , we denote by  $G^a$  the graph obtained from  $G$  by successively deleting any vertex  $x_i$  with  $a_i = 0$  and duplicating  $a_i - 1$  times any vertex  $x_i$  if  $a_i \geq 1$  (cf. [9, p. 53]). The notion of a parallelization was used in [6, 15] to describe the symbolic Rees algebra of an edge ideal. This notion has its origin in combinatorial optimization and has been used to describe the max-flow min-cut property of clutters [5, 22].

**Example 2.2.** Let  $G$  be the graph of Fig. 1 and let  $a = (3, 3)$ . We set  $x_i^1 = x_i$  for  $i = 1, 2$ . The parallelization  $G^a$  is a complete bipartite graph with bipartition  $V_1 = \{x_1^1, x_1^2, x_1^3\}$  and  $V_2 = \{x_2^1, x_2^2, x_2^3\}$ . Note that  $x_i^k$  is a vertex, i.e.,  $k$  is an index not an exponent.

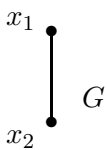


Fig. 1. Graph

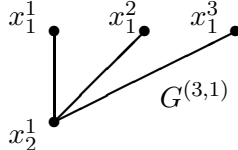


Fig. 2. Duplications of  $x_1$

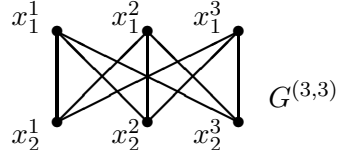


Fig. 3. Duplications of  $x_1$  and  $x_2$

**Definition 2.3.** Two edges of  $G$  are *independent* if they do not intersect. A *matching* of  $G$  is a set of pairwise independent edges. The *matching number* of  $G$ , denoted by  $\nu(G)$ , is the size of any maximum matching of  $G$ . A matching that covers all the vertices of  $V(G)$  is called a *perfect matching* of  $G$ .

A very readable and comprehensive reference about matchings in finite graphs is the book of Lovász and Plummer [14].

Given a graph  $G$ , the *edge-subring* of  $G$  is the subring  $K[G] = K[x_i x_j \mid \{x_i, x_j\} \in E(G)]$ .

**Lemma 2.4.** *Let  $G$  be a graph with vertex set  $X = \{x_1, \dots, x_n\}$  and let  $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ . Then  $G^a$  has a perfect matching if and only if  $x^a \in K[G]$ .*

*Proof.* We may assume that  $a_i \geq 1$  for all  $i$ , because if  $a$  has zero entries we can use the induced subgraph on the vertex set  $\{x_i \mid a_i > 0\}$ . The vertex set of  $G^a$  is

$$X^a = \{x_1^1, \dots, x_1^{a_1}, \dots, x_i^1, \dots, x_i^{a_i}, \dots, x_n^1, \dots, x_n^{a_n}\}$$

and the edges of  $G^a$  are exactly those pairs of the form  $\{x_i^{k_i}, x_j^{k_j}\}$  with  $i \neq j$ ,  $k_i \leq a_i$ ,  $k_j \leq a_j$ , for some edge  $\{x_i, x_j\}$  of  $G$ . We can regard  $x^a$  as an ordered multiset

$$x^a = x_1^{a_1} \cdots x_n^{a_n} = \underbrace{(x_1 \cdots x_1)}_{a_1} \cdots \underbrace{(x_n \cdots x_n)}_{a_n}$$

on the set  $X$ , that is, we can identify the monomial  $x^a$  with the multiset

$$X_a = \{\underbrace{x_1, \dots, x_1}_{a_1}, \dots, \underbrace{x_n, \dots, x_n}_{a_n}\}$$

on  $X$  in which each variable is uniquely identified with an integer between 1 and  $|a|$ . This integer is the position, from left to right, of  $x_i$  in  $X_a$ . There is a bijective map

$$\begin{array}{cccccccccccc} 1 & 2 & \cdots & a_1 & a_1 + 1 & \cdots & a_1 + a_2 & \cdots & a_1 + \cdots + a_{n-1} + 1 & \cdots & a_1 + \cdots + a_n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\ x_1 & x_1 & \cdots & x_1 & x_2 & \cdots & x_2 & \cdots & x_n & \cdots & x_n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\ x_1^1 & x_1^2 & \cdots & x_1^{a_1} & x_2^1 & \cdots & x_2^{a_2} & \cdots & x_n^1 & \cdots & x_n^{a_n} \end{array}$$

Hence if  $G^a$  has a perfect matching, then the perfect matching induces a factorization of  $x^a$  in which each factor corresponds to an edge of  $G$ , i.e.,  $x^a \in K[G]$ . Conversely, if  $x^a \in K[G]$  we can factor  $x^a$  as a product of monomials corresponding to edges of  $G$  and this factorization induces a perfect matching of  $G^a$ .  $\square$

Note that the process of passing from the vertex set  $X$  to the set  $X^a$  and the multiset  $X_a$  used in the lemma above can also be used to view a general monomial as a square-free monomial in a polynomial ring with additional variables. This is referred to as *polarization* in the literature. The copies of  $x_i$  that are used are called shadows of  $x_i$ . Conversely, a square-free monomial  $M$  in the ring  $K[X^a]$  can be viewed as a monomial in the ring  $K[X]$  by setting the exponent of  $x_i$  to be the number of shadows of  $x_i$  that divide  $M$ . This process is called *depolarization*.

Given an edge  $f = \{x_i, x_j\}$  of a graph  $G$ , we denote by  $G^f$  or  $G^{\{x_i, x_j\}}$  the graph obtained from  $G$  by successively duplicating the vertices  $x_i$  and  $x_j$ , i.e.,  $G^f := G^{1+e_i+e_j}$ , where  $e_i$  is the  $i$ th unit vector in  $\mathbb{R}^n$  and  $\mathbf{1} = (1, \dots, 1)$ .

**Example 2.5.** Consider the graph  $G$  of Fig. 4, where vertices are labeled with  $i$  instead of  $x_i$ . The duplication of the vertices  $x_1$  and  $x_2$  of  $G$  is shown in Fig. 6.

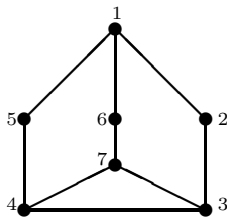


Fig. 4.  $G$

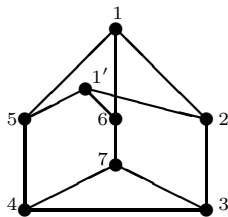


Fig. 5.  $G^{(2,1,1,1,1,1,1)}$

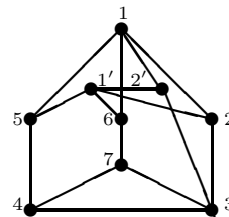


Fig. 6.  $G^{(2,2,1,1,1,1,1)} = G^{\{x_1, x_2\}}$

Recall that  $\text{def}(G)$ , the *deficiency* of  $G$ , is given by  $\text{def}(G) = |V(G)| - 2\nu(G)$ , where  $\nu(G)$  is the matching number of  $G$ . Hence  $\text{def}(G)$  is the number of vertices left uncovered by any maximum matching.

**Lemma 2.6.** *Let  $G$  be a graph and let  $a \in \mathbb{N}^n$  and  $c \in \mathbb{N}^q$ . Then*

- (a)  $x^a = x^\delta f^c$ , where  $|\delta| = \text{def}(G^a)$  and  $|c| = \nu(G^a)$ .
- (b)  $x^a$  belongs to  $I(G)^k \setminus I(G)^{k+1}$  if and only if  $k = \nu(G^a)$ .
- (c)  $(G^a)^f = (G^a)^{\{x_i, x_j\}}$  for any edge  $f = \{x_i^{k_i}, x_j^{k_j}\}$  of  $G^a$ .

*Proof.* Parts (a) and (b) follow using the bijective map used in the proof of Lemma 2.4. To show (c) we use the notation used in the proof of Lemma 2.4. We now prove the inclusion  $E((G^a)^f) \subset E((G^a)^{\{x_i, x_j\}})$ . Let  $y_i$  and  $y_j$  be the duplications of  $x_i^{k_i}$  and  $x_j^{k_j}$  respectively. We also denote the duplications of  $x_i$  and  $x_j$  by  $y_i$  and  $y_j$  respectively. The common vertex set of  $(G^a)^f$  and  $(G^a)^{\{x_i, x_j\}}$  is  $V(G^a) \cup \{y_i, y_j\}$ . Let  $e$  be an edge of  $(G^a)^f$ . If  $e = \{y_i, y_j\}$  or  $e \cap \{y_i, y_j\} = \emptyset$ , then clearly  $e$  is an edge of  $(G^a)^{\{x_i, x_j\}}$ . Thus, we may assume that  $e = \{y_i, x_\ell^{k_\ell}\}$ . Then  $\{x_i^{k_i}, x_\ell^{k_\ell}\} \in E(G^a)$ , so  $\{x_i, x_\ell\} \in E(G)$ . Hence  $\{x_i, x_\ell^{k_\ell}\}$  is in  $E(G^a)$ , so  $e = \{y_i, x_\ell^{k_\ell}\}$  is an edge of  $(G^a)^{\{x_i, x_j\}}$ . This proves the inclusion “ $\subset$ ”. The other inclusion follows using similar arguments (arguing backwards).  $\square$

**Theorem 2.7.** (Berge; see [14, Theorem 3.1.14]) *Let  $G$  be a graph. Then*

$$\text{def}(G) = \max\{c_0(G \setminus S) - |S| \mid S \subset V(G)\},$$

where  $c_0(G)$  denotes the number of odd components (components with an odd number of vertices) of a graph  $G$ .

We come to the main combinatorial result of this section.

**Theorem 2.8.** *Let  $G$  be a graph. Then  $\text{def}(G^f) = \delta$  for all  $f \in E(G)$  if and only if  $\text{def}(G) = \delta$  and  $\nu(G^f) = \nu(G) + 1$  for all  $f \in E(G)$ .*

*Proof.* Assume that  $\text{def}(G^f) = \delta$  for all  $f \in E(G)$ . In general,  $\text{def}(G) \geq \text{def}(G^f)$  for any  $f \in E(G)$ . We proceed by contradiction. Assume that  $\text{def}(G) > \delta$ . Then, by Berge’s theorem, there is an  $S \subset V(G)$  such that  $c_0(G \setminus S) - |S| > \delta$ . We set  $r = c_0(G \setminus S)$  and  $s = |S|$ . Let  $H_1, \dots, H_r$  be the odd components of  $G \setminus S$ .

Case (I):  $|V(H_k)| \geq 2$  for some  $1 \leq k \leq r$ . Pick an edge  $f = \{x_i, x_j\}$  of  $H_k$ . Consider the parallelization  $H'_k$  obtained from  $H_k$  by duplicating the vertices  $x_i$  and  $x_j$ , i.e.,  $H'_k = H_k^f$ . The odd connected components of  $G^f \setminus S$  are  $H_1, H_2, \dots, H_{k-1}, H'_k, H_{k+1}, \dots, H_r$ . Thus

$$c_0(G^f \setminus S) - |S| > \delta = \text{def}(G^f).$$

This contradicts Berge’s theorem when applied to  $G^f$ .

Case (II):  $|V(H_k)| = 1$  for  $1 \leq k \leq r$ . Notice that in this case  $S \neq \emptyset$  because  $G$  has no isolated vertices. Pick  $f = \{x_i, x_j\}$  an edge of  $G$  with  $\{x_i\} = V(H_1)$  and  $x_j \in S$ . Let  $y_i$  and  $y_j$  be the duplications of  $x_i$  and  $x_j$  respectively. The odd components of  $G^f \setminus (S \cup \{y_j\})$  are  $H_1, \dots, H_r, \{y_i\}$ . Thus

$$c_0(G^f \setminus (S \cup \{y_j\})) - |S \cup \{y_j\}| = c_0(G \setminus S) - |S| > \delta = \text{def}(G^f).$$

This again contradicts Berge’s theorem when applied to  $G^f$ . Therefore  $\text{def}(G) = \text{def}(G^f)$  for all  $f \in E(G)$ . Consequently  $\nu(G^f) = \nu(G) + 1$  for all  $f \in E(G)$ . The converse follows readily using the definition of  $\text{def}(G)$  and  $\text{def}(G^f)$ .  $\square$

The result of Theorem 2.8 depends upon the deficiency of  $G^f$  being constant for all  $f$ . In general, the deficiencies of  $G$  and  $G^f$  need not be equal.

**Example 2.9.** Consider the graph  $G$  of Fig. 7, where vertices are labeled with  $i$  instead of  $x_i$ . The duplication of the vertices  $x_3$  and  $x_4$  of  $G$  is shown in Fig. 8.

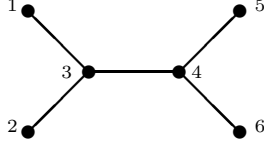


Fig. 7.  $\text{def}(G) = 2$

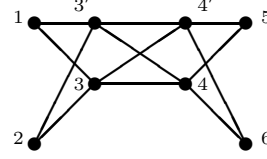


Fig. 8.  $\text{def}(G^{(1,1,2,2,1,1)}) = 0$

The theorem of Berge is equivalent to the following classical result of Tutte describing perfect matchings [14].

**Theorem 2.10.** (Tutte; see [4, Theorem 2.2.1]) *A graph  $G$  has a perfect matching if and only if  $c_0(G \setminus S) \leq |S|$  for all  $S \subset V(G)$ .*

We give the following characterization of perfect matchings in terms of duplications of edges.

**Corollary 2.11.** *Let  $G$  be a graph. Then  $G$  has a perfect matching if and only if  $G^f$  has a perfect matching for every edge  $f$  of  $G$ .*

*Proof.* Assume that  $G$  has a perfect matching. Let  $f_1, \dots, f_{n/2}$  be a set of edges of  $G$  that form a perfect matching of  $V(G)$ , where  $n$  is the number of vertices of  $G$ . If  $f = \{x_i, x_j\}$  is any edge of  $G$  and  $y_i, y_j$  are the duplications of the vertices  $x_i$  and  $x_j$  respectively, then clearly  $f_1, \dots, f_{n/2}, \{y_i, y_j\}$  form a perfect matching of  $V(G^f)$ . Conversely, if  $G^f$  has a perfect matching for all  $f \in E(G)$ , then  $\text{def}(G^f) = 0$  for all  $f \in E(G)$ . Hence, by Theorem 2.8, we get that  $\text{def}(G) = 0$ , so  $G$  has a perfect matching.  $\square$

The following lemma will play an important role in the proof of the main theorem. It uses the preceding combinatorial results about matchings to prove an algebraic equality.

**Lemma 2.12.** *Let  $I$  be the edge ideal of a graph  $G$ . Then  $(I^{k+1} : I) = I^k$  for  $k \geq 1$ .*

*Proof.* Let  $F = \{f_1, \dots, f_q\}$  be the set of all monomials  $x_i x_j$  such that  $\{x_i, x_j\} \in E(G)$ . Given  $c = (c_i) \in \mathbb{N}^q$ , we set  $f^c = f_1^{c_1} \cdots f_q^{c_q}$ . It is well known that the colon ideal of two monomial ideals is a monomial ideal, see for instance [26, p. 137]. In particular  $(I^{k+1} : I)$  is a monomial ideal. Clearly  $I^k \subset (I^{k+1} : I)$ . To show the reverse inclusion it suffices to show that any monomial of  $(I^{k+1} : I)$  is in  $I^k$ . Take  $x^a \in (I^{k+1} : I)$ . Then  $f_i x^a \in I^{k+1}$  for  $i = 1, \dots, q$ . We may assume that  $f_i x^a \notin I^{k+2}$ , otherwise  $x^a \in I^k$  as required. Thus  $x^{a+e_i+e_j} \in I^{k+1} \setminus I^{k+2}$  for any  $e_i + e_j$  such that  $\{x_i, x_j\} \in E(G)$ . Hence, by Lemma 2.6(b),  $\nu(G^{a+e_i+e_j}) = k+1$  for any  $\{x_i, x_j\} \in E(G)$ , that is,  $(G^a)^{\{x_i, x_j\}}$  has a maximum matching of size  $k+1$  for any edge  $\{x_i, x_j\}$  of  $G$ . With the notation used in the proof of Lemma 2.4, for any edge  $\{x_i^{k_i}, x_j^{k_j}\}$  of  $G^a$  we have

$$(G^a)^{\{x_i^{k_i}, x_j^{k_j}\}} = (G^a)^{\{x_i, x_j\}},$$

see Lemma 2.6(c). Then,  $(G^a)^f$  has a maximum matching of size  $k+1$  for any edge  $f$  of  $G^a$ . As a consequence

$$\text{def}((G^a)^f) = (|a| + 2) - 2(k+1) = |a| - 2k$$

for any edge  $f$  of  $G^a$ . Therefore, by Theorem 2.8,  $\text{def}(G^a) = |a| - 2k$ . Using Lemma 2.6(a), we can write  $x^a = x^\delta f^c$ , where  $|\delta| = \text{def}(G^a)$  and  $|c| = \nu(G^a)$ . Taking degrees in the equality  $x^a = x^\delta f^c$  gives  $|a| = |\delta| + 2|c| = (|a| - 2k) + 2|c|$ , that is,  $|c| = k$ . Then  $x^a \in I^k$  and the proof is complete.  $\square$

**Proposition 2.13.** *Let  $I = I(G)$  be the edge ideal of a graph  $G$  and let  $\mathfrak{m} = (x_1, \dots, x_n)$ . If  $\mathfrak{m} \in \text{Ass}(R/I^k)$ , then  $\mathfrak{m} \in \text{Ass}(R/I^{k+1})$ .*

*Proof.* As  $\mathfrak{m}$  is an associated prime of  $R/I^k$ , there is  $x^a \notin I^k$  such that  $\mathfrak{m}x^a \subset I^k$ . By Lemma 2.12 there is an edge  $\{x_i, x_j\}$  of  $G$  such that  $x_i x_j x^a \notin I^{k+1}$ . Then,  $x_\ell(x_i x_j x^a) \in I^{k+1}$  for  $\ell = 1, \dots, n$ , that is,  $\mathfrak{m}$  is an associated prime of  $R/I^{k+1}$ .  $\square$

To generalize from the maximal ideal to arbitrary associated primes, we will use localization. Since this process frequently results in disjoint graphs, we first recall the following fact about associated primes.

**Lemma 2.14.** ([11, Lemma 3.4], see also [3, Lemma 2.1]) *Let  $I$  be a square-free monomial ideal in  $S = K[x_1, \dots, x_m, x_{m+1}, \dots, x_r]$  such that  $I = I_1 S + I_2 S$ , where  $I_1 \subset S_1 = K[x_1, \dots, x_m]$  and  $I_2 \subset S_2 = K[x_{m+1}, \dots, x_r]$ . Then  $\mathfrak{p} \in \text{Ass}(S/I^k)$  if and only if  $\mathfrak{p} = \mathfrak{p}_1 S + \mathfrak{p}_2 S$ , where  $\mathfrak{p}_1 \in \text{Ass}(S_1/I_1^{k_1})$  and  $\mathfrak{p}_2 \in \text{Ass}(S_2/I_2^{k_2})$  with  $(k_1 - 1) + (k_2 - 1) = k - 1$ .*

Note that this lemma easily generalizes to an ideal  $I = (I_1, \dots, I_s)$  where the  $I_i$  are square-free monomial ideals in disjoint sets of variables. Then  $\mathfrak{p} \in \text{Ass}(R/I^k)$  if and only if  $\mathfrak{p} = (\mathfrak{p}_1, \dots, \mathfrak{p}_s)$ , where  $\mathfrak{p}_i \in \text{Ass}(R/I_i^{k_i})$  with  $(k_1 - 1) + \dots + (k_s - 1) = k - 1$ .

Note that although  $\mathfrak{p}_i$  is an ideal of  $R$ , the generators of  $\mathfrak{p}_i$  will generate a prime ideal in any ring that contains those variables. We will abuse notation in the sequel by denoting the ideal generated by the generators of  $\mathfrak{p}_i$  in any other ring by  $\mathfrak{p}_i$  as well.

We come to the main algebraic result of this paper.

**Theorem 2.15.** *Let  $G$  be a graph and let  $I = I(G)$  be its edge ideal. Then*

$$\text{Ass}(R/I^k) \subset \text{Ass}(R/I^{k+1})$$

*for all  $k$ . That is, the sets of associated primes of the powers of  $I$  form an ascending chain.*

*Proof.* Recall that we are assuming that  $G$  has no isolated vertices. Let  $\mathfrak{p}$  be an associated prime of  $R/I^k$  and let  $\mathfrak{m} = (x_1, \dots, x_n)$  be the irrelevant maximal ideal of  $R$ . For simplicity of notation we may assume that  $\mathfrak{p} = (x_1, \dots, x_r)$ . Then, the set  $C = \{x_1, \dots, x_r\}$  is a vertex cover of  $G$ . By Proposition 2.13, we may assume that  $\mathfrak{p} \subsetneq \mathfrak{m}$ . Write  $I_{\mathfrak{p}} = (I_2, I_1)_{\mathfrak{p}}$ , where  $I_2$  is the ideal of  $R$  generated by all square-free monomials of degree two  $x_i x_j$  whose image, under the canonical map  $R \rightarrow R_{\mathfrak{p}}$ , is a minimal generator of  $I_{\mathfrak{p}}$ , and  $I_1$  is the prime ideal of  $R$  generated by all variables  $x_i$  whose image is a minimal generator of  $I_{\mathfrak{p}}$ , which correspond to the isolated vertices of the graph associated to  $I_{\mathfrak{p}}$ . The minimal generators of  $I_2$  and  $I_1$  lie in  $S = K[x_1, \dots, x_r]$ , and the two sets of variables occurring in the minimal generating sets of  $I_1$  and  $I_2$  (respectively) are disjoint and their union is  $C = \{x_1, \dots, x_r\}$ . If  $I_2 = (0)$ , then  $\mathfrak{p}$  is a minimal prime of  $I$  so it is an associated prime of  $R/I^{k+1}$ . Thus, we may assume  $I_2 \neq (0)$ . An important fact is that localization preserves associated primes, that is  $\mathfrak{p} \in \text{Ass}(R/I^k)$  if and only if  $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}(R_{\mathfrak{p}}/(I_{\mathfrak{p}}R_{\mathfrak{p}})^k)$ , see [16, p. 38]. Hence,  $\mathfrak{p}$  is in  $\text{Ass}(R/I^k)$  if and only if  $\mathfrak{p}$  is in  $\text{Ass}(R/(I_1, I_2)^k)$  if and only if  $\mathfrak{p}$  is in  $\text{Ass}(S/(I_1, I_2)^k)$ . By Proposition 2.13 and Lemma 2.14,  $\mathfrak{p}$  is an associated prime of  $S/(I_1, I_2)^{k+1}$ . Hence, we can argue backwards to conclude that  $\mathfrak{p}$  is an associated prime of  $R/I^{k+1}$ .  $\square$

**Remark 2.16.** Using Proposition 2.13 and Lemma 2.14, this result can also be shown by induction on the number of variables because localizing at  $\mathfrak{p} \subsetneq \mathfrak{m}$  yields the ideal  $(I_1, I_2)$  in a polynomial ring with fewer than  $n$  variables. Using induction may be useful to extend Theorem 2.15 to other classes of monomial ideals (for instance to edge ideals of clutters, see [15]). In the case where the conclusion of Lemma 2.12 holds for a class of square-free monomial ideals, this result immediately extends.

**Corollary 2.17.** *Let  $I$  be a square-free monomial ideal and suppose  $(I^{k+1} : I) = I^k$  for  $k \geq 1$ . Then the sets of associated primes of the powers of  $I$  form an ascending chain.*

*Proof.* As in Proposition 2.13, we first show that  $\mathfrak{m} \in \text{Ass}(R/I^k)$  implies  $\mathfrak{m} \in \text{Ass}(R/I^{k+1})$ . Assume  $\mathfrak{m} \in \text{Ass}(R/I^k)$ . Then there is a monomial  $x^a \notin I^k$  with  $x_i x^a \in I^k$  for all  $i$ . By the hypothesis,  $x^a \notin (I^{k+1} : I)$ , so there is a square-free monomial generator  $e$  of  $I$  (which can be viewed as the edge of a clutter associated to  $I$ ) with  $ex^a \notin I^{k+1}$ . But  $x_i ex^a = e(x_i x^a) \in I^{k+1}$  for all  $i$ , so  $\mathfrak{m} \in \text{Ass}(R/I^{k+1})$ .

Recall that since  $I$  is finitely generated,  $(I^{k+1} : I)_{\mathfrak{p}} = (I_{\mathfrak{p}}^{k+1} : I_{\mathfrak{p}})$ . Thus  $(I_{\mathfrak{p}}^{k+1} : I_{\mathfrak{p}}) = I_{\mathfrak{p}}^k$ . The remainder of the argument now follows from localization, as in the proof of Theorem 2.15, after noting that Lemma 2.14 applies to an arbitrary square-free monomial ideal.  $\square$

In [19, Question 4.16] it was asked if the sets  $\text{Ass}(R/I^k)$  form an ascending chain for all square-free monomial ideals  $I$ . Corollary 2.17 provides one possible approach for answering this question for some classes of square-free monomial ideals. However, this approach will not work for all square-free monomial ideals, as can be seen by the following example.

**Example 2.18.** Let  $R = \mathbb{Q}[x_1, \dots, x_6]$  and let  $I$  be the square-free monomial ideal

$$I = (x_1x_2x_5, x_1x_3x_4, x_1x_2x_6, x_1x_3x_6, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_6, x_3x_5x_6, x_4x_5x_6).$$

Using *Normaliz* [2] together with *Macaulay2* [10], it is seen that  $I$  is a non-normal ideal such that  $(I^2 : I) = I$  and  $(I^3 : I) \neq I^2$ . Nevertheless, it is not hard to see that the sets of associated primes of the powers of  $I$  form an ascending chain and that the index of stability of  $I$  is equal to 3.

It is also of interest to note that for square-free monomial ideals, knowing that the sets  $\text{Ass}(R/I^k)$  form an ascending chain immediately implies that the sets  $\text{Ass}(I^{k-1}/I^k)$  form an ascending chain as well. Thus we get the following corollary of Theorem 2.15. A similar corollary would follow from Corollary 2.17 as well.

**Corollary 2.19.** *Let  $I = I(G)$  be the edge ideal of a graph  $G$ , then  $\text{Ass}(I^{k-1}/I^k)$  form an ascending chain for  $k \geq 1$ .*

*Proof.* It follows from [19, Lemma 4.4] and Theorem 2.15.  $\square$

### 3. INTEGRAL CLOSURES AND STABLE SETS

As mentioned in the Introduction, the sets of associated primes of the integral closures of the powers of  $I$  are also known to form an ascending chain that stabilizes. In order to compare the stable sets of the two chains  $\text{Ass}(R/I^k)$  and  $\text{Ass}(R/\overline{I^k})$ , we recall the following definition and lemma.



**Definition 3.1.** Let  $I = (x^{v_1}, \dots, x^{v_q})$  be a monomial ideal of  $R$ . The *Rees algebra* of  $I$ , denoted by  $R[It]$ , is the monomial subring

$$R[It] = R[x^{v_1}t, \dots, x^{v_q}t] \subset R[t].$$

The ring  $\mathcal{F}(I) = R[It]/\mathfrak{m}R[It]$  is called the *special fiber ring* of  $I$ . The Krull dimension of  $\mathcal{F}(I)$ , denoted by  $\ell(I)$ , is called the *analytic spread* of  $I$ .

**Lemma 3.2.** [26, Proposition 7.1.17, Exercise 7.4.10] *Let  $I = (x^{v_1}, \dots, x^{v_q})$  be a monomial ideal and let  $A$  be the matrix with column vectors  $v_1, \dots, v_q$ . If  $\deg(x^{v_i}) = d$  for all  $i$ , then*

$$\mathcal{F}(I) \simeq K[x^{v_1}t, \dots, x^{v_q}t] \simeq K[x^{v_1}, \dots, x^{v_q}] \quad \text{and} \quad \ell(I) = \dim K[x^{v_1}, \dots, x^{v_q}] = \text{rank}(A).$$

Once again, localization will allow us to reduce to the case of the maximal ideal. To that end, we prove a result that characterizes when  $\mathfrak{m}$  is in the stable sets of  $\text{Ass}(R/I^k)$  and  $\text{Ass}(R/\overline{I^k})$ .

**Proposition 3.3.** *Let  $G$  be a graph. The following are equivalent:*

- (a)  $\mathfrak{m} \in \text{Ass}(R/I(G)^k)$  for some  $k$ .
- (b) The connected components of  $G$  are non-bipartite.
- (c)  $\mathfrak{m} \in \text{Ass}(R/\overline{I(G)^t})$  for some  $t$ .
- (d)  $\text{rank}(A) = n$ , where  $A$  is the incidence matrix of  $G$  and  $n = |V(G)|$ .

*Proof.* The equivalence between (c) and (d) follows from [17, Theorem 3] because the analytic spread of  $I$  is equal to the rank of  $A$ , see Lemma 3.2. The equivalence between (b) and (d) follows from the fact that  $\text{rank}(A) = |V(G)|$  if  $G$  is a connected non-bipartite graph and  $\text{rank}(A) = |V(G)| - 1$  if  $G$  is a connected bipartite graph, see [26, Lemma 8.3.2].

Let  $G_1, \dots, G_r$  be the connected components of  $G$ . We set  $S_i = K[V(G_i)]$  and  $\mathfrak{m}_i = (V(G_i))$ . Assume that  $\mathfrak{m} = (\mathfrak{m}_1, \dots, \mathfrak{m}_r)$  is an associated prime of  $R/I(G)^k$  for some  $k$ . Then, by Lemma 2.14, there are positive integers  $k_i$  such that  $\mathfrak{m}_i$  is an associated prime of  $S_i/I(G_i)^{k_i}$ . Therefore  $G_i$  is non-bipartite for all  $i$  because edge ideals of bipartite graphs are normally torsion-free [23]. This proves that (a) implies (b). Finally we prove that (b) implies (a). Assume that  $G_i$  is non-bipartite for all  $i$ . Then, by [3, Corollary 3.4],  $\mathfrak{m}_i \in \text{Ass}(S_i/I(G_i)^{k_i})$  for  $k_i \gg 0$ . Then, again by Lemma 2.14, it follows that  $\mathfrak{m}$  is an associated prime of  $R/I(G)^k$  for some  $k$ .  $\square$

Combining Lemma 3.2 with Proposition 3.3(d) illustrates the importance of the analytic spread in determining associated primes. Localizing will allow the use of these results for primes other than  $\mathfrak{m}$ , but this will require control over the analytic spread of the edge ideal of a disconnected graph.

**Lemma 3.4.** *Let  $L_1, L_2$  be monomial ideals with disjoint sets of variables. If  $L_1, L_2$  are generated by monomials of degrees  $d_1$  and  $d_2$  respectively, then  $\ell(L_1 + L_2) = \ell(L_1) + \ell(L_2)$ .*

*Proof.* We set  $L = L_1 + L_2$ . Let  $g_1, \dots, g_r$  and  $h_1, \dots, h_s$  be the minimal generating sets of  $L_1$  and  $L_2$  respectively that consist of monomials. By hypothesis  $L_1$  (resp.  $L_2$ ) lives in a polynomial ring  $K[\mathbf{x}]$  (resp.  $K[\mathbf{y}]$ ), where  $\mathbf{x} = \{x_1, \dots, x_q\}$  and  $\mathbf{y} = \{y_1, \dots, y_m\}$ . We set  $R = K[\mathbf{x}, \mathbf{y}]$ . The special fiber ring of  $L$  can be written as

$$\mathcal{F}(L) \simeq K[\mathbf{x}, \mathbf{y}, u_1, \dots, u_r, z_1, \dots, z_s]/(\mathbf{x}, \mathbf{y}, J),$$

where  $J$  is the presentation ideal of the Rees algebra  $R[Lt]$  and  $u_1, \dots, u_r, z_1, \dots, z_s$  is a set of new indeterminates. The ideal  $J$  is the kernel of the map

$$K[\mathbf{x}, \mathbf{y}, u_1, \dots, u_r, z_1, \dots, z_s] \rightarrow R[Lt], \quad x_i \mapsto x_i, y_j \mapsto y_j, u_i \mapsto g_i t, z_j \mapsto h_j t.$$

Since  $J$  is a toric ideal, there is a generating set of  $J$  consisting of binomials of the form

$$x^\alpha y^\beta u^\gamma z^\delta - x^{\alpha'} y^{\beta'} u^{\gamma'} z^{\delta'}$$

such that  $x^\alpha y^\beta g^\gamma h^\delta t^{|\gamma|+|\delta|} = x^{\alpha'} y^{\beta'} g^{\gamma'} h^{\delta'} t^{|\gamma'|+|\delta'|}$ . From this equation we get  $x^\alpha g^\gamma = x^{\alpha'} g^{\gamma'}$ ,  $y^\beta h^\delta = y^{\beta'} h^{\delta'}$  and  $t^{|\gamma|+|\delta|} = t^{|\gamma'|+|\delta'|}$ . Hence

$$|\alpha| + d_1|\gamma| = |\alpha'| + d_1|\gamma'|, |\beta| + d_2|\delta| = |\beta'| + d_2|\delta'|, |\gamma| + |\delta| = |\gamma'| + |\delta'|.$$

We claim that  $\deg(x^\alpha y^\beta) = 0$  if and only if  $\deg(x^{\alpha'} y^{\beta'}) = 0$ . Assume that  $\deg(x^\alpha y^\beta) = 0$ , i.e.,  $\alpha = \beta = 0$ . From the first equality we have  $|\alpha'| = d_1(|\gamma| - |\gamma'|)$ . From the second and third equality we get

$$|\beta'| + d_2|\delta'| = d_2|\delta| = d_2(|\gamma'| + |\delta'| - |\gamma|) \Rightarrow |\beta'| = d_2(|\gamma'| - |\gamma|).$$

As  $|\alpha'| \geq 0$  and  $|\beta'| \geq 0$ , we get  $\gamma - \gamma' = 0$ . Thus  $\alpha' = \beta' = 0$ . This proves the claim. Therefore one has the following simpler expression for the special fiber ring of  $L$

$$(3.1) \quad \mathcal{F}(L) \simeq K[u_1, \dots, u_r, z_1, \dots, z_s]/P \simeq K[g_1t, \dots, g_rt, h_1t, \dots, h_st],$$

where  $P$  is the toric ideal of  $K[g_1t, \dots, g_rt, h_1t, \dots, h_st]$ . Let  $A_1$  (resp.  $A_2$ ) be the matrix whose columns are the exponent vectors of the monomials  $g_1t, \dots, g_rt$  (resp.  $h_1t, \dots, h_st$ ) and let  $A$  be the matrix whose columns are the exponent vectors of  $g_1t, \dots, g_rt, h_1t, \dots, h_st$ . The sets of variables  $\mathbf{x}$  and  $\mathbf{y}$  are disjoint. Therefore  $\text{rank}(A) = \text{rank}(A_1) + \text{rank}(A_2)$ . Since

$$\mathcal{F}(L_1) = K[g_1t, \dots, g_rt] \quad \text{and} \quad \mathcal{F}(L_2) = K[h_1t, \dots, h_st],$$

using Lemma 3.2 and Eq. (3.1), it follows that  $\ell(L) = \ell(L_1) + \ell(L_2)$ .  $\square$

**Remark 3.5.** When  $L_2$  is generated by a set of variables, which is the case that we really need, the lemma follows at once from [18, Corollary 6.2, p. 43] because in this case the set of variables form an asymptotic sequence over  $L_1$  in the sense of [18].

The sets  $\text{Ass}(R/I^i)$  and  $\text{Ass}(R/\overline{I^i})$  stabilize for large  $i$ . The next result shows that, for edge ideals, their corresponding stable sets are equal.

**Theorem 3.6.** *Let  $I$  be the edge ideal of a graph  $G$ . There exists a positive integer  $N$  such that  $\text{Ass}(R/I^k) = \text{Ass}(R/\overline{I^k})$  for  $k \geq N$ .*

*Proof.* Recall that we are assuming that  $G$  has no isolated vertices. By [1] there is a positive integer  $N_1$  such that  $\text{Ass}(R/I^{N_1}) = \text{Ass}(R/I^k)$  for  $k \geq N_1$ , and by [20, 21], there is a positive integer  $N_2$  such that  $\text{Ass}(R/\overline{I^{N_2}}) = \text{Ass}(R/\overline{I^k})$  for  $k \geq N_2$ . Let  $N = \max\{N_1, N_2\}$ , and assume that  $k \geq N$ . First we show the inclusion “ $\subset$ ”. Take  $\mathfrak{p}$  in  $\text{Ass}(R/I^k)$ .

Case (I):  $\mathfrak{p} = \mathfrak{m}$ . By Proposition 3.3,  $\mathfrak{p} \in \text{Ass}(R/\overline{I^i})$  for some  $i$ . Hence  $\mathfrak{p} \in \text{Ass}(R/\overline{I^k})$  because the sets  $\text{Ass}(R/\overline{I^j})$  form an ascending sequence, see [18, Proposition 3.4, p. 13].

Case (II):  $\mathfrak{p} = (x_1, \dots, x_r) \subsetneq \mathfrak{m}$ . Let  $I_1, I_2$ , and  $S$  be as in the proof of Theorem 2.15 and let  $X_i$  be the set of variables that occur in the minimal generating set of  $I_i$ . Notice that  $\mathfrak{p} = (X_1, X_2)$ . As  $\mathfrak{p}$  is an associated prime of  $S/(I_1 + I_2)^k$ , applying Lemma 2.14 to  $I_1S + I_2S$ , where we regard  $I_i$  as an ideal of  $S_i = K[X_i]$ , we can write  $\mathfrak{p} = \mathfrak{p}_1S + \mathfrak{p}_2S$ , where  $\mathfrak{p}_1 \in \text{Ass}(S_1/I_1^{k_1})$  and  $\mathfrak{p}_2 \in \text{Ass}(S_2/I_2^{k_2})$ , with  $(k_1 - 1) + (k_2 - 1) = k - 1$ . Notice that  $\mathfrak{p}_i = (X_i)$ . Thus, applying Proposition 3.3 to the graph  $G_2$  associated to  $I_2$ , we get that the rank of the incidence matrix  $A_{G_2}$  of  $G_2$  is  $|X_2|$ . On the other hand  $\ell(I_2)$ , the analytic spread of  $I_2$ , is equal to the Krull dimension of the edge subring  $K[G_2]$ , which is equal to the rank of  $A_{G_2}$  (see Lemma 3.2). Since  $I_1$  is generated by  $|X_1|$  variables, one has  $\ell(I_1) = |X_1|$ . By Lemma 3.4, the analytic spread  $\ell(I_1 + I_2)$  is equal to  $|X_1| + |X_2| = \text{ht}(\mathfrak{p})$ . Thus, using [17, Theorem 3], we conclude that  $\mathfrak{p} \in S/(\overline{I_1 + I_2})^i$

for  $i \gg 0$ . Then,  $\mathfrak{p}R_{\mathfrak{p}} \in R_{\mathfrak{p}}/\overline{(I_1 + I_2)}_{\mathfrak{p}}^i = R_{\mathfrak{p}}/\overline{I_{\mathfrak{p}}}^i$ . Consequently, by [16, Corollary, p. 38] and the fact that the integral closure of ideals commute with localizations, we get  $\mathfrak{p} \in R/\overline{I}^k$ .

The inclusion “ $\supset$ ” holds for any ideal  $I$  of a commutative Noetherian ring  $R$  by a result of Ratliff [20, 21], see [18, Proposition 3.17] for additional details.  $\square$

**Corollary 3.7.** *Let  $G$  be a graph and let  $I$  be its edge ideal. Then  $I$  is normally torsion-free if and only if  $\text{Ass}(R/\overline{I}^i) = \text{Ass}(R/I)$  for  $i \geq 1$ .*

*Proof.*  $\Rightarrow$ ) This implication follows at once by noticing that  $I$  is normal, i.e.,  $\overline{I}^i = I^i$  for  $i \geq 1$ .

$\Leftarrow$ ) Since  $\text{Ass}(R/I) \subset \text{Ass}(R/I^i)$  for  $i \geq 1$ , it suffices to show that  $\text{Ass}(R/I^i) \subset \text{Ass}(R/I)$  for  $i \geq 1$ . Let  $\mathfrak{p}$  be an associated prime of  $R/I^i$  and let  $N$  be the index of stability of  $I$ . Then, by Theorem 2.15,  $\mathfrak{p}$  is an associated prime of  $R/I^N$ . Hence, by Theorem 3.6,  $\mathfrak{p}$  is an associated prime of  $R/\overline{I}^k$  for  $k \gg 0$ . Thus by hypothesis  $\mathfrak{p}$  is an associated prime of  $I$ .  $\square$

**Procedure 3.8.** The following simple procedure for *Macaulay2* (version 1.4) decides whether  $\text{Ass}(R/I^3)$  is contained in  $\text{Ass}(R/I^4)$  and whether we have the equality  $\text{Ass}(R/I^3) = \text{Ass}(R/I^4)$ . It also computes  $\overline{I}^4$  and decides whether  $\text{Ass}(R/I^4)$  is equal to  $\text{Ass}(R/\overline{I}^4)$ .

```
R=QQ[x1,x2,x3,x4,x5,x6,x7,x8,x9];
load "normaliz.m2";
I=monomialIdeal(x1*x2,x2*x3,x1*x3,x3*x4,x4*x5,x5*x6,x6*x7,x7*x8,x8*x9,x5*x9);
isSubset(ass(I^3),ass(I^4))
ass(I^3)==ass(I^4)
(intCl4,normRees4)=intclMonIdeal I^4;
intCl4'=substitute(intCl4,R);
ass(monomialIdeal(intCl4'))==ass(I^4)
```

The next example was computed using version 1.4 of *Macaulay2* [10]. This version allows the use of *Normaliz* [2] inside *Macaulay2* in order to compute the integral closure of a monomial ideal and the normalization of the Rees algebra of a monomial ideal. Example 3.9 shows that although the stable sets of  $\text{Ass}(R/I^i)$  and  $\text{Ass}(R/\overline{I}^i)$  are equal, they do not need to be reached at the same power.

**Example 3.9.** Let  $R = \mathbb{Q}[x_1, \dots, x_9]$  and let  $I = I(G)$  be the edge ideal of the graph below (Fig. 9). Notice that this example was computed without using Theorem 3.6.

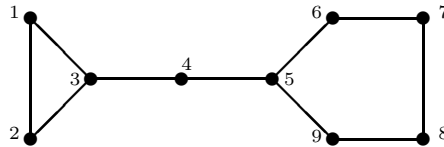


Fig. 9. Graph  $G$  with non-normal  $I(G)$

Using *Macaulay2* (see Procedure 3.8), together with the fact that the index of stability of  $I$  is at most 8 [3, Corollary 4.3] and the fact that the stable set of  $\text{Ass}(R/\overline{I}^i)$  is contained in the

stable set of  $\text{Ass}(R/I^i)$  [18, Proposition 3.17], we get

$$\begin{aligned} I^i &= \overline{I}^i, \quad i = 1, 2, 3, \quad \overline{I}^4 = I^4 + (x^a), \quad \overline{I}^5 = I^5 + I(x^a), \quad \text{where } x^a = x_1x_2x_3x_5x_6x_7x_8x_9, \\ \text{Ass}(R/I^i) &= \text{Ass}(R/\overline{I}^i) \text{ for } i \neq 4 \text{ and } \text{Ass}(R/\overline{I}^4) \subsetneq \text{Ass}(R/I^4), \\ \text{Ass}(R/I) &\subsetneq \text{Ass}(R/I^2) \subsetneq \text{Ass}(R/I^3) \subsetneq \text{Ass}(R/I^4) = \text{Ass}(R/I^i) \text{ for } i \geq 4, \\ \text{Ass}(R/\overline{I}^3) &\subsetneq \text{Ass}(R/\overline{I}^4) \subsetneq \text{Ass}(R/\overline{I}^5) = \text{Ass}(R/\overline{I}^i) \text{ for } i \geq 5. \end{aligned}$$

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